The Language of ∞-categories

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Talk (1) **The language of ∞-categories (60 minutes)**

Give an informal introduction to ∞-categories as categories in which morphisms may be composed only up to coherent homotopy. Briefly mention quasi-categories and the homotopy-coherent nerve of a simplicial category, but do not spend much time on their technical definitions.

Introduce the mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ and the homotopy category $h\mathcal{C}$. Explain how equivalences and subcategories of an ∞-category are defined. Discuss the principal examples: the ∞-category of spaces $\mathcal{S}$, the ∞-category Cat_∞ of small ∞-categories, and localizations of ordinary or enriched categories at weak equivalences.

Introduce functor ∞-categories $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Define limits and colimits by their universal properties. Similarly, explain adjunctions between ∞-categories, emphasizing their similarity to ordinary categorical adjunctions. Discuss the presheaf category $\mathrm{PSh}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ and the Yoneda embedding $\mathcal{C} \to \mathrm{PSh}(\mathcal{C})$. Explain restriction along a functor and its left and right adjoints, given by left and right Kan extension. State the pointwise formulas for Kan extensions, at least informally, and mention the naturality of the Yoneda embedding with respect to left Kan extension. Introduce over- and undercategories and explain that their mapping spaces can be described as fibers of maps between mapping spaces. Give a conceptual account of straightening and unstraightening: functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ correspond to Cartesian fibrations over $\mathcal{C}$. As the main example, explain the equivalence $\mathrm{PSh}(X) \simeq \mathcal{S}/X$ for a space X.

Time permitting, mention that limits of ∞-categories have mapping spaces computed as limits of mapping spaces.

References: [Lur09, Lan21, Hau23, Hau25]

1 INTRO

First off my name is RILEY MORISS, from MELBOURNE. Ok well since Im giving the first talk and we havent met each other yet and I made it here without any real knowledge of ∞-categories, I want to TAKE A QUICK POLL on how comfortable people are with these things; ZERO KNOWLEDGE, SOME EXPERIENCE, EXPERT. Ill try to tailor things a bit to the audience but I was given a very long list of things to cover, so THIS WILL BE TERSE. Finally I was told that young researchers tend to appologise too much when they give talks, so I WILL APOLOGISE ONLY ONCE, if its too slow, too fast, I get something wrong or there are questions please interrupt.

2 WHAT IS AN INFINITY CATEGORY

I guess the goal of the talk is to DEFINE AN ∞-CATEGORY OF ∞-CATEGORIES, and then describe how to do category theory within such a thing. The first thing to do then is give some DESIDERATA, pieces of the elephant that we wish to sketch, as it where. (Un)fortunately there are many ways to motivate and go about this. We will choose a single one, that of category theory.

In ORDINARY CATEGORY THEORY we have objects and morphisms between them. The only 'real' restriction on these things is that the composition of MORPHISMS BE UNITAL AND ASSOCIATIVE. This turns out to still be too restrictive to capture many structures that appear naturally in mathematics. The most elementary example I know where this happens is in the WEAK TWO CATEGORY OF SPANS. Another prominent example is the STRICT TWO CATEGORY OF (SMALL) CATEGORIES. Because we only want to think of objets up to isomorphism in category theory an equivalence of categories should be functors that are mutually inverse *up to isomorphism*. This produces the familiar notion of categorical equivalence and makes the collection of categories more naturally a two category.

Thus we would like an enrichment of the ideas of ordinary category theory that naturally accommodate these structures, without going all the way to set theory where we dont have any real tools. Thus the 'idea' of an ∞-category is that it is a category where composition of morphisms are no longer required to be associative. Instead we require that there be an extra structure that compares the different associations, namely AN ISOMORPHISM. To make sense of this however we would need between any two *morphisms* a collection of morphisms; apply this idea recursively, we require that the isomorphism, defined by having a two sided inverse, only have an inverse *up to isomorphism* and that composition in these morphism sets itself only be associative up to higher maps.

IDEA: A CATEGORY BUT INSTEAD OF ASSOCIATIVE COMPOSITION WE HAVE THAT THE MAPPING SPACES ARE CATEGORIES AND ASSOCIATION IS ONLY UP TO ISOMORPHISM; RECURSE. WHAT SHOULD MORPHISMS BETWEEN THEM BE THOUGH...

2.1 MODELS

This idea has been made rigeous in the setting of (one) category theory through several different 'models', that is constructions of verifiable mathematical entites. The different models are things like quasi-categories, simplicially enriched categories, (complete) Segal spaces, Segal categories and relative categories. THESE PROVIDE A DEFINITION OF A SINGLE ∞-CATEGORY, THE ∞-CATEGORY OF ∞-CATEGORIES WILL NEED TO BE CONSTRUCTED SEPARATELY. Note that these things are models for $(\infty, 1)$-categories, ∞-categories in which all $n \geq 2$ simplicies are isomorphisms.

TOPOLOGICAL CATEGORIES. Recall the Quillen equivalence

$$|-| : \mathrm{sSet} \rightleftarrows \mathrm{Top} : \mathrm{Sing}$$

where sSet has a model structure defined by 'pulling back' along the model structure on Top along the geometric realisation (literally for the weak equivalences, but not-literally for fibrations). THIS

IS DIFFERENT FROM THE JOYAL MODEL STRUCTURE. IN PARTICULAR IN THE FORMER THE FIBRANT OBJECTS ARE KAN COMPLEXES, WHILE IN THE LATTER IT IS QUASI-CATEGORIES. Note that Sing's essential image is ∞-groupoids. Thus we use the terms Kan complex, ∞-groupoid and space interchangebly.

A TOPOLOGICALLY ENRICHED CATEGORIES or simply a TOPOLOGICAL CATEGORY. For topological spaces we already know how to construct an infinite tower of maps between maps, namely homotopies and so we can use these structures to define our categories in which we will have maps between maps etc. A simplicially enriched category, via our adjunction above will allow us to do the same thing.

Definition (Def 1.1.1.6, [Lur09])**.** *A topological category is a category enriched over (compactly generated Hausdorff) spaces. In particular between any objects of the category X, Y, Z there is a space $\mathrm{Hom}(X, Y)$ and the composition map*

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$$

is continuous.

Now its a bit odd that this is allowed to be a strict category, werent we trying to avoid strict associativity? Well it turns out that it doesnt matter, if we bothered to write out a 'weakly' associative topological category (for instance using quasi-categories) it would turn out that all of them where equivalent (as ∞-categories) to one of this form. LURIE ASSURES ME OF THIS, I THINK ITS IN SECTION 1 AND 2. So where does this leave us, we can trade in our study of weak associativity for strict associativity with the extra structure of spaces. This turns out to be much more difficult however, as one has to smash everything with rigidification results.

SO AN INFINITY CATEGORY CAN BE BOTH A SPACE AND A CATEGORY ENRICHED IN SPACES, VERY CONFUSING STUFF.

QUASI-CATEGORIES.

Definition. *The simplex category has objects the natural numbers (≥ 1) and morphisms weakly monotonic maps (the elements of natural numbers are other natural numbers). A* SIMPLICIAL SET *is a presheaf on the simplex category, they form a category with natural transformations as morphisms. A simplicial set is called an ∞-category if all inner horns fill and an ∞-groupoid (or Kan complex) if all horns fill (roughly a map from the top edges of the simplex extend to maps over the whole simplex).*

Intuitively a simplical set should be though of as assigning to the zero simplex a set of objects, to the one simplex a set of morphisms, *and to the higher simplicies the collections of possible compositions of the morphisms*. THESE POSSSIBLE COMPOSITIONS CAN BE CONFLATED WITH THE HIGHER MORPHISMS IN A PROCESS WE DISCUSS LATER. The point is that one now has all the data of 'possible ways of composing maps' and the functoriality (and filling conditions) say that these compositions behave in the sort of weakly associative way we wanted.

A FUNCTOR OF INFINITY CATEGORIES IS SIMPLY A MAP OF SIMPLICIAL SETS, one of the main advantages of this model. In this case we have already a full subcategory of sSet, and we would like to observe that the mapping sets are naturally *simplicially* enriched. sSet is naturally enriched over itself in the following way. Let $X_\bullet, Y_\bullet$ be simplicial sets, then define

$$\mathrm{Hom}_{\mathrm{sSet}}(X_\bullet, Y_\bullet)_n \quad := \quad \mathrm{Hom}_{\mathrm{sSet}}(X_\bullet \times \Delta^n, Y_\bullet).$$

For two infinity categories $\mathcal{C}, \mathcal{D}$ it is a fact that this internal hom is itself an infinity categroy that we denote $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

COHERENT NERVE. If these are both models then there should be a construction that sends one to the other. Indeed this is the 'coherent nerve', which will send a topological category to a simplicial set, in a way that encodes the topological structure (rather than just taking the nerve). The moral of the homotopy coherent nerve (hcn) is to look at all n-simplicies of the category that have an $n+1$ simplex whitnessing the commutativity of the n simplex (which will be a diagram).

Let $[n] = \{0, 1, ..., n\}$, be a poset category. Then this category can be topologically enriched by defining

$$\mathrm{Hom}_{[n]}(i,j) \quad := \quad \begin{cases} [0,1]^{j-i-1}, & i < j \\ *, & i = j \\ , & i > j \end{cases}$$

Notice there are 2^{j-i-1} verticies of the cube $[0,1]^{j-i-1}$, they correspond to *subsets* of the *coordinates of the cube*, which are indexing the *intermediate objects between the two objects* (the path is a path between two things). This topological category is denoted Path$[n]$.

Let $\mathcal{C}$ be a topologically enriched category, then

$$[n] \mapsto \mathrm{Hom}_{\mathrm{top-cat}}(\mathrm{Path}[n], \mathcal{C})$$

is a simplicial set that we denote $\mathrm{N}^{hc}_{\bullet}(\mathcal{C})$. IF YOU WANT TO DO THINGS SIMPLICIALLY WHEREVER YOU SEE A PATH YOU SHOULD THINK OF IT AS THE ONE SIMPLEX OF A SIMPLICIAL SET $\Delta^n \to X$. This coherent nerve has an adjoint $\mathfrak{C}$. FIND THE LAND REFERENCE It is a fact that if the simplicial category is locally Kan, that is the mapping simplicial sets are in fact mapping spaces then the coherent nerve returns an infinity category.

2.2 Mapping spaces.

So let $\mathcal{C}$ be an infinity category and $x, y \in \mathcal{C}$ be two objects, either elements of $\mathcal{C}[0]$ or for a topological category, just objects. In a topological category it is clear what the mapping spaces $\mathcal{C}(x, y)$ are between two these objects, however for a quasi-cateogry it is less clear.

We have the collection of one simplicies joining the two vertices however this should really be topologised, taking into account the higher simplicies right? We could apply the right adjoint to the coherent nerve, but this is a cheat. First we should also note the term, *mapping spaces*, if we have an $(\infty, 1)$-category then the morphism set, if it forms an ∞-category should have one simplicies that are 'basically' the two simplicies of the original quasi-cateogy, in particular all the one simplicies of the mapping simplicial set, should be invertible, exactly making it an ∞-groupoid. By the homotopy hypothesis these are exactly spaces and thus we obtain the connection to the topological categories case.

RIGHT MAPPING SPACES. [Lur09, Prop 1.2.2.3] defines the right mapping spaces $\mathrm{Hom}_{\mathcal{C}}(x, y)$, with the n-simplicies to be the $n+1$-simplicies of $\mathcal{C}$ such that the last vertex Δ^{n+1} is constant on y and the subsimplex spanned by the first n verticies is constant on x.

Example. *Let $n = 2$ in the mapping space. Then we have a solid tetrahedron $n = 3$ in the ambiet simplicial set. A whole face is mapped to x, geometrically crushed. What remains is two verticies, x, y with three edges between them, that is filled in. There is therefore a space of paths and then an extra dimension for homotopies of paths, exactly a two simplex.*

So from a quasi-category we can obtain a simplicial category in this way, the objects are the zero simplicies and the mapping spaces are as constructed. This agrees, up to homotopy with the coherent nerve constructions.

2.3 going back to one categories

From now on we will let $\mathcal{C}$ be an ∞-category, we have explained three'ish models and will now act like the way they are connected is so canonical that nothing could be more obvious. We can get an ordinary category out of a topological category in two ways, either forget the topology or mod out by equivalences. The latter is called THE HOMOTOPY CATEGORY $h\mathcal{C}$. It is an ordinary one category with the same objects as $\mathcal{C}$ and the morphism *sets* are now

$$\mathrm{Hom}_{h\mathcal{C}}(X,Y) \quad := \quad \pi_0 \, \mathrm{Hom}_{\mathcal{C}}(X,Y)$$

Again for quasi-categories apply the adjoint to the coherent nerve and then take π_0.

2.4 EXAMPLES

2.4.1 SMALL ∞-CATEGORIES

We started with an infinity category of functors, this is a genuine infinity category (not a Kan complex) and comes from a construction *in simplicial sets*. To get a simplicial enrichment of quasi-cateogories such that the coherent nerve will be an infinity category we need to ensure that the mapping simplicial sets are Kan complexes and so we take the CORE of these functor categories, the largest Kan complex contained in it. This is A DIFFERENT SIMPLICIAL ENRICHMENT! To this we apply the coherent nerve to get Cat_∞. From this perspective it is completely clear that the mapping space in the ∞-category of ∞-categories is the core of the functor categories.

2.4.2 LOCALISATIONS AT WEAK EQUIVALENCES

Relative categories are categories equipped with a class of weak equivalences $(\mathcal{C}, W)$, this class of weak equivalences is require to be a subcategory (clased under compositions) and also 'wide' which means that all objects are in the subcategory (meaning that all identity morphisms are weak equivalences). One can 'localise' this category in two ways, one categorically and ∞-categorically. It is easier to understand both the classical and infinity categorical localisations via universal properties. One categorically the localisation is a functor category denotes $L : \mathcal{C} \to \mathcal{C}[W^{-1}]$ such that if $F : \mathcal{C} \to \mathcal{D}$ is a functor such that all elements of W are mapped to isomorphisms in $\mathcal{D}$ then there is a unique map making the diagram commute

$$\begin{tikzcd} \mathcal{C} \arrow[r, "L"] \arrow[dr, "F"'] & \mathcal{C}[W^{-1}] \arrow[d, dashed, "\exists!"] \\ & \mathcal{D} \end{tikzcd}$$

Infinity categorically we probably want to replace the uniqueness condition with a contractable condition. [Hau, Prop 5.7.4] So let $L : \mathcal{C} \to \mathcal{C}'$ be a functor between ∞-categories, that sends a collectino W of morphisms in $\mathcal{C}$ to equivalences in $\mathcal{C}'$ (an equivalence in $\mathcal{C}'$ is a morphism such that its image in the homotopy category is an isomorphism). Then L exhibits $\mathcal{C}'$ as a localistaion $\mathcal{C}[W^{-1}]$ iff for any other ∞-category $\mathcal{D}$ composition with L is a fully-faithful functor,

$$L^* : \mathrm{Fun}(\mathcal{C}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

whose ESSENTIAL? REFERENCE DOESNT USE IT BUT? image is those functors that send maps in W to equivalences. WE WILL DEFINE THAT PROPERLY IN THE NEXT SECTION. We can see that structurally this mirrors the one categorical case, now requiring that if we have a map that inverts weak equivalences, say a given $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ that its fiber / preimage under L^*, the space of 'factorisations' is contractable (fully-faithful has contractable fibers). THE IDEA IS LITERALLY REPLACE UNIQUE WITH CONTRACTABLE, BUT TO ACTUALLY PHRASE WHAT SPACE IS CONTRACTABLE AND MAKE IT RIGOUROUS REQUIRES SOME LABOUR.

The ∞-category can be denoted $\mathcal{C}[W^{-1}]$, or for extra clarity $N(\mathcal{C})[W^{-1}]$. It is a fact that

$$hN(\mathcal{C})[W^{-1}] \simeq \mathcal{C}[W^{-1}]$$

That is the classical localisation is the homotopy category of the ∞-categorical localisation.

SPACES The category of spaces with weak homotopy equivalences. Using the model of topologically enriched categories we can see immediately that the category of (cgh) topological spaces is enriched over itself using the topology of Steenrod. THIS IS NOT HOWEVER 'THE' INFINITY CATEGORY OF SPACES. THE category of spaces is defined as the localisation of Top at weak homotopy equivalances, and we denote this infinity category $\mathcal{S}$. To get something equivalent to this we can also restrict to the subcategory of CGH on the bifibrant objects, namely the CW complexes, which is a topological category that is equivalent.

Given any ∞-category $\mathcal{C}$ then we have another infinity category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, which is denoted $\mathrm{PSh}(\mathcal{C})$, the ∞-cateogry of presheaves on $\mathcal{C}$. TO UNDERSTAND OP THINK IN THE TOPOLOGICAL CATEGORY MODEL.

CHAIN COMPLEXES The category of chain complexes of R-modules with quasi-isomorphisms, which defines the derived category $\mathcal{D}(R)$.

SPECTRA The category of spectra with weak equivalences (a map of spectra that induces an iso on all homotopy groups), which gives Sp.

3 BASIC CATEGORY THEORY WITH ∞-CATEGORIES

From now on everything is ∞labeled so we drop it unless explicitly noted other wise.

3.1 EQUIVALENCES AND SUBCATEGORIES

Kerodon 4.8

(FULLY) FAITHFUL. A functor $F : \mathcal{C} \to \mathcal{D}$ is faithful if if for all objects $x, y \in \mathcal{C}$ the induced map

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(Fx, Fy)$$

is a homotopy equivelance onto a connected component Def 1.2.1.1. [Lan21, Def 2.3.2] It is fully-faithful if for all pairs

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(Fx, Fy)$$

is a homotopy equivalence.

SUB-CATEGORY. If we think of an infinity category as a generalised notion of space then the notion that SUBCATEGORIES TRIES TO CAPTURE IS THAT OF THE INCLUSION OF A CONNECTED COMPONENT. If we are in the quasi-category model then, there are two things that need to be dealt with to ensure that a sub-simplicial set is a sub-∞-category, WE NEED TO ENSURE THAT THE SIMPLICIAL SET IS ITSELF AN INFINITY CATEGORY, AND WE NEED TO ENSURE THAT IT DOES NOT CHANGE THE SHAPE. Thus we define a sub-∞-category $\mathcal{C}' \subseteq \mathcal{C}$ as a sub-simplicial set (subset on the images of the functor) that is itself an ∞-category *and* the inclusion $\mathcal{C}' \to \mathcal{C}$ is faithful LURIE USES THAT THE MAP IS AN INNER FIBRATION, HOWEVER BECAUSE ITS A SUB-SIMPLICIAL, MONO-IN sSet, THE MAP ON THE KAN COMPLEXES WILL ALSO BE INJECTIVE STRICTLY AND HENCE HAVE EMPTY OR CONTRACTABLE FIBERS, WHICH IS THE SAME AS AN INCLUSION OF A CONNECTED COMPONENT, HENCE FAITHFUL..

ESSENTIALLY SURJECTIVE. [Lan21, Def 2.3.3] A functor as above is essentially surjective if the induced map on the homotopy category is an essential surjection.

EQUIVALENCE. [Hau, Cor 2.7.6] An equivalence is a fully faithful and essentially surjective functor.

3.2 (CO)LIMITS

[Hau, §4.1, §5.2] I will write speak only of limits and leave it as an exercise to turn around all the arrows. Classically a limit is a universal cone, that is a cone such that there is a unique map from any other cone, ∞-categorically we will replace the notion of unique with *contractable*.

An object in an ∞-category $x \in \mathcal{C}$ is called TERMINAL if for all $c \in \mathcal{C}$ the mapping space $\mathcal{C}(c, x)$ is contractable.

First we require the over category for an object $c \in \mathcal{C}$, denoted $\mathcal{C}_{/c}$, which is the PULLBACK (OF SIMPLICAL SETS, NOT INFINITY CATEGORIES) of the span

$$\{c\} \to \mathcal{C} \xleftarrow{\mathrm{ev}_1} \mathrm{Fun}([1], \mathcal{C})$$

Intuitively the objects ARE PAIRS, AN OBJECT OF $\mathcal{C}$ AND A MAP TO C, THE MAPS ARE MAPS OVER C. In contrast to the ordinary case, the maps over C, the triangles, however need only commute up to homotopy. [Lan21, Prop 3.3.18] Let $f : x \to z, g : y \to z$ be objects of $\mathcal{C}_{/z}$ then there is a homotopy caretesian diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}_{/z}}(f,g) \arrow[r, dashed] \arrow[d, dashed] & \mathrm{Hom}_{\mathcal{C}}(x,y) \arrow[d, "g_*"] \\
* \arrow[r, "f"] & \mathrm{Hom}_{\mathcal{C}}(x,z)
\end{tikzcd}$$

In particular the mapping space in the over category is the homotopy fiber of pushing forward by g.

Now let $p : \mathcal{J} \to \mathcal{C}$ be a functor then THE ∞-CATEGORY OF CONES OF p is given by the pullback AGAIN AS SIMPLICIAL SETS, of the span

$$\mathcal{C} \to \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \leftarrow \mathrm{Fun}(\mathcal{J}, \mathcal{C})_{/p}.$$

The first arrow is simply $c \mapsto (J \mapsto c)$, the constant functor on the specified element; the second arrow is the arrow coming from the pullback diagram defining the over category. A LIMIT OF p IS THEN A TERMINAL OBJECT IN THIS CATEGORY.

ALTERNATE DEF. What we describe above is a definition via the universal property of the terminal objects and slice categories. There is another characterisation, loosely described by Lurie. Given a functor $F : \mathcal{C} \to \mathcal{D}$ then we say $Y \in \mathcal{D}$ is a limit of F if there is a natural transformation $\alpha : \underline{Y} \to F$ such that for every $X \in \mathcal{D}$ composition with α induces a homotopy equivalence

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(\underline{X}, F).$$

$\underline{X}, \underline{Y}$ are the constant functors on those objects. So maps into a limit are the same as natural transformations to the functor.

3.3 ADJUNCTIONS

Lure 5.2.2.8 [Hau] 6.3.

Let $f : \mathcal{C} \to \mathcal{D} : g$ be a pair of functors between ∞-categories. Then f is the left adjoint to g iff there is a natural transformation $\mathrm{id}_{\mathcal{C}} \to g \circ f$ such that for every pair of objects $C \in \mathcal{C}, D \in \mathcal{D}$ we have that the composition

$$\mathrm{Hom}_{\mathcal{D}}(f(C), D) \xrightarrow{g} \mathrm{Hom}_{\mathcal{C}}(g(f(C)), g(D)) \xrightarrow{-\circ u(C)} \mathrm{Hom}_{\mathcal{C}}(C, g(D))$$

is a homotopy equivalence.

To contrast this with the one categorical case, it looks like exactly the natural 'bijection' between $\mathrm{Hom}(F(-), -)$ and $\mathrm{Hom}(-, G(-)$, however the existence is not asserted but provided as a structure; namely the structure of the unit. Thus it is sort of 'between' the two one categorical analogues. One asserts the existence of an natural bijection, the other provides two natural transformations satisfying a property, that induce a natural bijection. ∞-categorically we require a structure that induces a natural bijection; it is the first definition replaced with a strucutre that inherintly appears as the second definition.

4 SLIGHTLY LESS BASIC CATEGORY THEORY WITH ∞-CATEGORIES

4.1 Kan extensions

Now we get into things that I DONT REALLY UNDERSTAND EVEN ONE CATEGORICALLY.

4.1.1 restriction along a functor

[Lur09, 4.3.3]. Lurie defines Kan extensions via a mapping cylinder. He notes the following characterisations however

Lemma (Prop 4.3.3.7)**.** *Let $\delta : K^0 \to K^1$ be a map of simplicial sets and $\mathcal{D}$ be an ∞-category. δ induces a restriction functor, via precomposition*

$$\delta^* : \mathrm{Fun}(K^1, \mathcal{D}) \to \mathrm{Fun}(K^0, \mathcal{D})$$

If δ^* has a left adjoint $\delta_!$ then we call it a functor of left Kan extensions along δ.

So the image of a single functor under this adjiont will be its Kan extension along δ.

4.1.2 pointwise formulas

So if we extend is it possible to actually compute the new functors values on some object? Yes, by taking a colimit over the original functors values. This is possible in the general setting [Hau, §8] but more complicated, so we will state it in a special case. Consider then when $\delta : \mathcal{C}_0 \to \mathcal{C}$ is an inclusion of a full subcategory. Then for $F \in \mathrm{Fun}(\mathcal{C}_0, \mathcal{D})$ [Lur09, Proof of Lem 4.3.2.13]

$$\delta_!(F)(C) \simeq \mathrm{colim}\big(\mathcal{C}_{0/C} \to \mathcal{C}_0 \xrightarrow{F} \mathcal{D}\big).$$

We can see that (left) Kan extensions are built from colimits and so a sufficient condition for Kan extensions to exists is that $\mathcal{D}$ be co-complete.

Let $\delta : K^0 \to K^1$ be a map of simplicial sets and $\mathcal{D}$ an ∞-category for which the left Kan extension $\delta_! : \mathrm{Fun}(K^0, \mathcal{D}) \to \mathrm{Fun}(K^1, \mathcal{D})$ is defined. In general, the terminology "Kan extension" is perhaps somewhat unfortunate: if $F : K^0 \to \mathcal{D}$ is a diagram, then $\delta^*\delta_! F$ need not be equivalent to F. If δ is fully faithful, then the unit map $F \to \delta^*\delta_! F$ is an equivalence: this follows from Proposition 4.3.3.5. We will later need the following more precise assertion:

Proposition 4.3.3.8. *Let $\delta : \mathcal{C}^0 \to \mathcal{C}^1$ and $f_0 : \mathcal{C}^0 \to \mathcal{D}$ be functors between ∞-categories and let $f_1 : \mathcal{C}^1 \to \mathcal{D}$, $\alpha : f_0 \to \delta^* f_1 = f_1 \circ \delta$ be a left Kan*

4.2 (un)Straightening

Consider a map of sets $f : X \to Y$, then at each $y \in Y$ there is a fiber $X_y = f^{-1}(y)$. The fibers form a Y indexed family of sets, a map $Y \to \mathrm{Set}$, mapping $y \mapsto X_y$, moreover it is clear that $X = \bigcup_{y \in Y} X_y$, which in essence defines an inverse from the functor to the map to Y. This construction is functorial and defines an equivalence of one categories.

$$\mathrm{Set}_{/Y} \simeq \mathrm{Fun}(Y, \mathrm{Set})$$

Intuitively a fibration of simplicial sets is one which the fibers are ∞-groupoids and depend functorially on the assignment. A cartesian fibration is the same except instead of the fibers being ∞-groupoids they are merely required to be ∞-categories, the cartesian, right, left, inner, etc are all just different lifting properties.

In the infinity category world we have the following anallogous results, given by equivilences of categories. From most general to least general, [Lan21, Thm 3.3.10, 3.3.17, 3.3.16] let $\mathcal{C}$ be an ∞-category, then

$$\mathrm{Cart}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty)$$

$$\mathrm{RFib}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{PSh}(\mathcal{C})$$

If $\mathcal{C}$ is an ∞-groupoid, or a space, we change to the notation X and we have further

$$\mathrm{RFib}(X) \simeq \mathcal{S}_{/X} \simeq \mathrm{PSh}(X).$$

STRAIGHTENING is specifically the arrow $\mathcal{S}_{/X} \to \mathrm{PSh}(X)$, and unstraightening its inverse.

EXAMPLE: YONEDA [Hau, Prop 4.4.1] to prove Yoneda one shows that some map $\mathcal{C} \to \mathrm{RFib}(\mathcal{C}$ is fully-faithful and therefore by straightening we have the standard statement: The Yoneda embedding $\mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ is fully-faithful.

Classically the Yoneda embedding sends $C \in \mathcal{C}$ to the presheaf $\mathrm{Hom}_{\mathcal{C}}(-, C)$, and this is exactly mirrored ∞-categorically, where now the hom set is a hom ∞-groupoid, noting that our presheaves are of spaces now and not of sets.

If we consider a morphism $f : \mathcal{C} \to \mathcal{D}$, then we get an induced map, given by the Yoneda functor, $\mathrm{PSh}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{D})$, but we also have that $y_{\mathcal{C}} : \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ is fully faithful. Thus we can Kan extend $y_{\mathcal{D}} \circ f$ along the Yoneda functor, which [Lur09, Lemma 5.2.6.7] agrees with the Yoneda functor of f. i.e. the following commutes

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d, "y_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "y_{\mathcal{D}}"] \\
\mathrm{PSh}(\mathcal{C}) \arrow[r, "(y_{\mathcal{C}})_!(y_{\mathcal{D}} \circ f)"'] & \mathrm{PSh}(\mathcal{D})
\end{tikzcd}$$

References

[Hau] Rune Haugseng. YET ANOTHER INTRODUCTION TO -CATEGORIES.

[Lan21] Markus Land. *Introduction to Infinity-Categories*. Compact Textbooks in Mathematics. Springer International Publishing, Cham, 2021.

[Lur09] Jacob Lurie. *Higher topos theory*. Number no. 170 in Annals of mathematics studies. Princeton University Press, Princeton, N.J, 2009. OCLC: ocn244702012.